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Joint probability distributions for disordered 1D wires

P J Roberts

The Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BZ, UK

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Abstract. The scattering of a one-dimensional disordered wire consisting of elastic, time-reversal-invariant scatterers, is specified completely by the transmission intensity T, reflection phase ϕ_r and transmission phase ϕ_t. The problem of extracting the joint distribution $p_{1\to Lz}(T,\phi_r,\phi_t)$ of these variables for a system of large length Lz, given the distribution $p_1(T,\phi_r,\phi_t)$ of the individual scatterers is examined without recourse to the random-phase assumption and without restriction to weak disorder. The method adopted is to expand the distribution in terms of irreducible representations of SU(1,1), the group topologically formed by the k-space transfer matrices which describe the multiple scattering. Both the bulk of the distribution, and the resonance tail are examined. The method is applied to the case of the Anderson model over a range of disorders.

1. Introduction

It is well known that in the absence of inelastic effects, the resulting coherent interference of the electronic wavefunction in a disordered system induces strong fluctuations. Quantities such as the conductance become sensitive to the precise impurity arrangement, and the probability distributions describing such quantities can become broader as the system-size is increased. For one-dimensional and quasi one-dimensional disordered wires consisting of elastic scatterers, it has been established [1–9] that the logarithm of the conductance is a well behaved statistical quantity, being described approximately by a Gaussian for long lengths, and that this variable should be used to characterize the ensemble as opposed to the non-self-averaging conductance. Particular emphasis has been put on the scaling properties of the distribution; specifically, the question arises as to whether single-parameter scaling is obeyed. It is by now well established [7, 8] that, for 1D systems, the distribution is characterized by a single parameter only in the limit of weak disorder.

Despite the log-normal form describing the bulk of the distribution of the conductance of long 1D wires, deviations occur in the tails, and statistical averages using this form must be taken with care. In particular, the moments $\langle g^{\nu}\rangle$ ($\nu > 1/2$) of the conductance are never given correctly by such an average [8, 10, 11]; these moments are dominated by statistically rare resonance-states which give rise to exceptionally large conductances.

Theoretical studies of the conductance have been based on the Landauer formula [12] which relates the conductance to the transmission intensity T of the disordered system, thus placing particular interest on the scattering characteristics of the wire.

The case of random-phase models has been studied in particular detail by Mello [7], who derives an analytic expression for both the mean and the variance of the distribution in terms of the distribution of the individual scatterers (assumed statistically independent but equivalent). For such ensembles, the distributions of the transmission and reflection phases (which together with the transmission intensity completely characterize the scattering of a single-channel system consisting of time-reversal invariant scatterers) are taken as flat over their range. This assumption induces great simplifications in the description of the evolution of the probability distribution with system-size, and extensions of analytic work to more physical models have concentrated on the limit of weak disorder [8]. For non-random-phase models, there also arises the non-trivial question of the form attained by the transmission and reflection phase distributions in the limit of large system lengths [13–16].

In this paper, the asymptotic joint probability distribution of the transmission intensity (T), reflection phase (ϕ_r) and transmission phase (ϕ_t) is derived for ensembles not restricted to being comprised of random-phase scatterers nor to the limit of weak disorder. The approach is based, as in [7], on the observation that the k-space transfer matrices (defined below), describing single-channel elastic scatterers obeying time-reversal symmetry, topologically form the group SU(1,1). The unitary irreducible representations of this group which form a complete set are known [17], and direct expansion of the joint probability distribution in terms of these representations facilitates the calculation of its form for large system lengths. Both the bulk of the distribution and the resonance-tail can be extracted. The use of representation theory in the study of disordered systems was initiated by Kirkman and Pendry in the extraction of the long-length behaviour of the moments $\langle T^\nu \rangle$ ($\nu > 1/2$) [11], and in the calculation of the localization length and density of states [18] (see also [19] for such calculations). The asymptotic distribution of the reflection phase has also been studied by such means [15]. In references [11, 15, 18, 19], the representations are referred to as 'generalized transfer matrices'.

The transfer matrix provides a simple means with which to describe multiple scattering; it relates the incident and reflected wave amplitudes at the left to those at the right of a scattering system. In terms of the reflection and transmission coefficients (r and t respectively) of the medium, it has the explicit form

$$\mathbf{M} = \begin{bmatrix} t^{-1} & t^{-1}r \\ t^{*-1}r^* & t^{*-1} \end{bmatrix}. \tag{1}$$

The transfer matrix ($\mathbf{M}_{1\to Lz}$) for a system composed of Lz scatterers can be constructed from the the transfer matrices of the individual units by simple multiplication

$$\mathbf{M}_{1\to Lz} = \prod_{n=1}^{Lz} \mathbf{M}_n. \tag{2}$$

The fact that the transfer matrices of the scatterers are all members of the group SU(1,1), and that the right-hand side of (2) is just a product of such elements, allows powerful group-theoretical results to be applied in the calculation of the system-length dependence of quantities of interest. The next section discusses the application of such ideas to the extraction of the joint probability distribution.

2. Methodology

Cast in terms of transfer matrices, it is required to obtain, given the distribution $p_1(\mathsf{M})$ of the individual scatterers, the probability distribution $p_{1\to Lz}(\mathsf{M})$ resulting from the successive 'convolutions' implied by (2) (the scatterers are assumed statistically independent but equivalent). The approach taken here will be to directly expand the probability density $p_{1\to Lz}(\mathsf{M})$ in terms of the irreducible representations of SU(1,1), the group formed by the (k-space) transfer matrices. A discussion of the properties of these irreducible representations that will be required is presented in the appendix.

With the notation of the appendix, the spectral analysis takes the form [7]

$$p_{1\to Lz}(\mathsf{M}) = \sum_{D^{\pm}} \sum_{\substack{k=1,3/2,\dots \\ mm'}} D^{k}_{mm'}(\mathsf{M})(\langle D^{k}\rangle^{*}_{Lz})_{mm'} w_k + \sum_{C^{0,1/2}} \sum_{mm'} \int D^{1/2+is}_{mm'}(\mathsf{M})(\langle D^{1/2+is}\rangle^{*}_{Lz})_{mm'} w(s)\,\mathrm{d}s \tag{3}$$

where the following definition has been made

$$(\langle D^{k}\rangle_{Lz})_{mm'} = \int D^{k}_{mm'}(\mathsf{M}) p_{1\to Lz}(\mathsf{M})\,\mathrm{d}\mu(\mathsf{M}). \tag{4}$$

The probability density is here considered to be relative to the invariant measure $\mathrm{d}\mu(\mathsf{M})$, so that in particular the normalization condition is $\int p_{1\to Lz}(\mathsf{M})\,\mathrm{d}\mu(\mathsf{M}) = 1$. Exploiting the statistical independence and equivalence of the individual scatterers, and utilizing the representation property

$$D^{k}(\mathsf{M}_{1\to Lz}) = \prod_{n=1}^{Lz} D^{k}(\mathsf{M}_n) \tag{5}$$

equation (4) can be re-expressed as

$$(\langle D^{k}\rangle_{Lz})_{mm'} = (\langle D^{k}\rangle^{Lz})_{mm'} \tag{6}$$

where the average is taken over the probability distribution for a single scatterer.

The procedure will be to extract the contributions in the expansion (3) which dominate for large system lengths. The case of the bulk of the distribution will be examined first.

3. The bulk of the distribution

It is convenient to determine the contributions of the various classes of representations separately. An examination of the contribution of the continuous class C^0 will be performed first.

3.1. Contribution of the class C^0

It is required to evaluate, for large Lz, the set of integrals

$$I^{C^0}_{mm'}(\mathsf{M}, Lz) = \int_0^\infty D^{1/2+\mathrm{i}s}_{mm'}(\mathsf{M})(\langle D^{1/2+\mathrm{i}s}\rangle^{*Lz})_{mm'} w(s)\,\mathrm{d}s$$
$$m, m' = 0, \pm1, \pm2, \ldots . \tag{7}$$

The case of random-phase models has been examined in detail by Mello [7]. For such ensembles, the averaged representation $\langle D^{1/2+\mathrm{i}s}\rangle$ has zero entries except for the element $m = 0, m' = 0$, so that the matrix exponentiation in (7) can be performed trivially. Only the integral $I^{C^0}_{00}$ is non-vanishing. Furthermore, the averaged representations in the classes $C^{1/2}$, $D^{\pm}$ vanish identically, implying that the entire joint probability distribution is given simply by $I^{C^0}_{00}(\mathsf{M}, Lz)$. This integral can be performed [7] for large Lz by the saddle-point method.

For non-circular ensembles, the averaged representations do not condense, implying in particular that the matrix exponentiation is no longer straightforward. In the evaluation of (7), it will be found necessary to deform the contour of integration, so that it is required to obtain the analytic continuation of $\langle D^{1/2+\mathrm{i}s}\rangle^{*}_{Lz}$ into the complex plane of s. Such a continuation has been accomplished by Kirkman and Pendry [11] for the set of representations $\hat{\chi}^{2(-1/2-\mathrm{i}s)}$ which are related to the $D^{1/2+\mathrm{i}s} \in C^0$ by a simple unitary similarity transformation (equation (A.10)). It is shown in [11] that the $\hat{\chi}^{2(-1/2-\mathrm{i}s)}$ retain their representation status for *arbitrary* s, allowing the replacement $\langle\hat{\chi}^{2(-1/2-\mathrm{i}s)}\rangle_{Lz} = \langle\hat{\chi}^{2(-1/2-\mathrm{i}s)}\rangle^{Lz}$ at any complex s and implying that

$$(\langle\hat{\chi}^{2(-1/2-\mathrm{i}s)}\rangle^{*Lz})_{\mathrm{continuation}} = \langle\hat{\chi}^{2(-1/2-\mathrm{i}s)*}_{\mathrm{continuation}}\rangle^{Lz}. \tag{8}$$

Although the similarity transformation relation between $D^{1/2+\mathrm{i}s}$ and $\hat{\chi}^{2(-1/2-\mathrm{i}s)}$ can also be extended to complex s, it is found to be more convenient to span the class C^0 with the $\hat{\chi}^{2(-1/2-\mathrm{i}s)}$ due to the absence of square-root factors. In terms of these representations, one has

$$I^{C^0}_{-m-m'}(\mathsf{M}, Lz) = \int_0^\infty \hat{\chi}^{2(-1/2-\mathrm{i}s)}_{mm'}(\mathsf{M})(\langle\hat{\chi}^{2(-1/2-\mathrm{i}s)*}\rangle^{Lz})_{mm'} w(s)\,\mathrm{d}s. \tag{9}$$

Henceforth in this section, the parametrization μ, T, ν will be understood (where μ and ν are related to ϕ_{r} and ϕ_{t} by equation (A.1)).

In order to describe the bulk of the distribution for large Lz, it is permissible to use the asymptotic forms (for $T \ll 1$) of the elements $\hat{\chi}^{2(-1/2-\mathrm{i}s)}_{mm'}(\mathsf{M})$. For compactness in the display of explicit forms, it will be assumed that $m \geqslant m'$ (the $I^{C^0}_{-m-m'}$ for $m < m'$ can then be constructed by use of the identity $I^{C^0}_{mm'}(\mathsf{M}, Lz) = I^{C^0 *}_{-m-m'}(\mathsf{M}, Lz)$). With this restriction, one has (over the region of complex s which will be of interest)

$$\hat{\chi}^{2(-1/2-\mathrm{i}s)}_{mm'}(\mathsf{M}) = \exp[\mathrm{i}(2m\mu + 2m'\nu)]\frac{\Gamma\left(\frac{1}{2} - \mathrm{i}s + m\right)}{\Gamma\left(\frac{1}{2} - \mathrm{i}s + m'\right)}$$
$$\times \left[T^{1/2-\mathrm{i}s}\frac{\Gamma\left(\frac{1}{2} + \mathrm{i}s\right)\Gamma(1 + \mathrm{i}s)2^{2\mathrm{i}s}}{2\pi^{1/2}\mathrm{i}s\Gamma\left(\frac{1}{2} + \mathrm{i}s + m\right)\Gamma\left(\frac{1}{2} + \mathrm{i}s - m'\right)}\right.$$
$$\left. + T^{\frac{1}{2}+\mathrm{i}s}\frac{\Gamma\left(\frac{1}{2} - \mathrm{i}s\right)\Gamma(1 - \mathrm{i}s)2^{-2\mathrm{i}s}}{-2\pi^{1/2}\mathrm{i}s\Gamma\left(\frac{1}{2} - \mathrm{i}s + m\right)\Gamma\left(\frac{1}{2} - \mathrm{i}s - m'\right)}\right]. \tag{10}$$

The $\{I^{C^0}_{mm'}(\mathsf{M}, Lz)\}$ are then given by

$$I^{C^0}_{-m-m'}(\mathsf{M}, Lz) = \exp[\mathrm{i}(2m\mu + 2m'\nu)]T^{1/2} \times \int_0^\infty \{\exp[\eta_{mm'}(s) + \xi_{mm'}(s) - \mathrm{i}s\ln T] + \exp[\eta_{mm'}(-s) + \xi_{mm'}(s) + \mathrm{i}s\ln T]\}\,\mathrm{d}s \tag{11}$$

with

$$\exp[\eta_{mm'}(s)] = \frac{\Gamma\left(\frac{1}{2}+\mathrm{i}s\right)\Gamma(1+\mathrm{i}s)2^{2\mathrm{i}s}\tanh\pi s}{i\pi^{1/2}\Gamma\left(\frac{1}{2}+\mathrm{i}s+m\right)\Gamma\left(\frac{1}{2}+\mathrm{i}s-m'\right)} \tag{12}$$

and

$$\exp[\xi_{mm'}(s)] = \frac{\Gamma\left(\frac{1}{2}-\mathrm{i}s+m\right)}{\Gamma\left(\frac{1}{2}-\mathrm{i}s+m'\right)}(\langle\hat{\chi}^{2(-1/2-\mathrm{i}s)*}\rangle^{Lz})_{mm'}. \tag{13}$$

The $\hat{\chi}^{2(-1/2+\mathrm{i}s)}(\mathsf{M})$ have the property [11]

$$\hat{\chi}^{2(-1/2+\mathrm{i}s)}_{mm'}(\mathsf{M}) = \frac{\Gamma\left(\frac{1}{2}+\mathrm{i}s+m\right)\Gamma\left(\frac{1}{2}-\mathrm{i}s+m'\right)}{\Gamma\left(\frac{1}{2}-\mathrm{i}s+m\right)\Gamma\left(\frac{1}{2}+\mathrm{i}s+m'\right)}\hat{\chi}^{2(-1/2-\mathrm{i}s)}_{mm'}(\mathsf{M}). \tag{14}$$

This symmetry implies that $\xi_{mm'}(s)$ is a function of s^2, so that

$$I^{C^0}_{-m-m'}(\mathsf{M}, Lz) = \exp[\mathrm{i}(2m\mu + 2m'\nu)]T^{1/2}\int_{-\infty}^{\infty}\exp[\omega_{mm'}(s)]\,\mathrm{d}s \tag{15}$$

with

$$\omega_{mm'}(s) = \eta_{mm'}(s) + \xi_{mm'}(s) - \mathrm{i}s\ln T. \tag{16}$$

The further symmetry

$$\chi^{2(-1/2-\mathrm{i}s)*}_{mm'}(\mathsf{M}) = \chi^{2(-1/2+\mathrm{i}s)}_{-m-m'}(\mathsf{M}) \qquad \text{real } s \tag{17}$$

together with (14) allows the $\{\xi_{mm'}(s)\}$ to be expressed in a form which is more convenient for analytic continuation:

$$\exp[\xi_{mm'}(s)] = \frac{\Gamma\left(\frac{1}{2}+\mathrm{i}s+m\right)}{\Gamma\left(\frac{1}{2}+\mathrm{i}s+m'\right)}(\langle\hat{\chi}^{2(-1/2-\mathrm{i}s)}\rangle^{Lz})_{-m-m'}. \tag{18}$$

To proceed further, it is necessary to examine the behaviour of the $\{\omega_{mm'}(s)\}$ in the complex plane of s in the hope that an integration contour can be found that will facilitate the evaluation of the integrals $\{I^{C^0}_{mm'}(\mathsf{M}, Lz)\}$. In this connection, it will transpire that the region close to $s = \mathrm{i}/2$ is of particular interest. At the value $s = \mathrm{i}/2$, $\hat{\chi}^{2(-1/2-\mathrm{i}s)} = \hat{\chi}^{2(0)}$ attains the form

$$\langle\hat{\chi}^{-2(0)}\rangle = \begin{bmatrix} \hat{\chi}^{2(0)}_- & |a_-\rangle & \mathbf{0} \\ 0\ldots0 & 1 & 0\ldots0 \\ \mathbf{0} & |a_+\rangle & \hat{\chi}^{2(0)}_+ \end{bmatrix} \tag{19}$$

where the (infinite) matrices $\hat{\chi}_{\pm}^{2(0)}$, and vectors $|a_{\pm}\rangle$ have non-trivial dependence on the parameters of the model. The eigenspectrum of this decomposable form comprises a single eigenvalue at unity, and the union of the eigenspectra of the matrices $\hat{\chi}_{\pm}^{2(0)}$. It is easily shown that, except in special cases, the latter two subspaces have upper spectrum-limits (in modulus) which lie below unity. The exceptions correspond to situations such as a periodic system with no disorder, or the band-centre of the Anderson model with pure off-diagonal disorder, and will not be considered here. This situation of a discrete eigenvalue at the top of the spectrum will subsist at least in the region close to $s = \mathrm{i}/2$, and it is convenient within this region to decompose $\langle \hat{\chi}^{2(-1/2-\mathrm{i}s)} \rangle$ into the sum of the contribution of the discrete eigenvalue and of the remaining part of the spectrum

$$\langle \hat{\chi}^{2(-1/2-\mathrm{i}s)} \rangle = |\lambda^{(0)}(s)\rangle \exp[\psi(s)] \langle \lambda^{(0)}(s)| + \langle \hat{\chi}^{2(-1/2-\mathrm{i}s)} \rangle^{(+\,-)}. \tag{20}$$

In the above equation, the eigenvalue of greatest modulus has been denoted $\exp[\psi(s)]$ for later convenience. It will also be useful to perform a corresponding decomposition of the $\{\omega_{mm'}(s)\}$,

$$\omega_{mm'}(s) = \omega_{mm'}^{(0)}(s) + \omega_{mm'}^{(+\,-)}(s) \tag{21}$$

with superscript (0) pertaining to the eigenvalue of greatest modulus and $(+\,-)$ to the remainder of the spectrum.

The particular case of $I_{00}^{C^0}$, from which the probability distribution of the single variable T is immediately obtained as $p(T)\mathrm{d}T = I_{00}^{C^0}(T, Lz)\,\mathrm{d}T/T^2$, will now be investigated. For large Lz, and for T attaining a value within the bulk of the distribution, it is found that $\omega_{00}^{(0)}(s)$ possesses a saddle-point which is close to $s = \mathrm{i}/2$. The contour of integration will be deformed to pass through this saddle, allowing the contribution of the eigenvalue of greatest modulus to be determined by the method of steepest-descents. The saddle-point analysis proceeds in a manner which is precisely analogous to that which is presented in [7] for random-phase models, and yields the following contribution to $I_{00}^{C^0}$:

$$I_{00}^{(0)C^0}(\mathrm{M}, Lz) = \frac{1}{2}\left[\frac{2\pi}{|\psi''(s_a)|Lz}\right]^{1/2}\frac{1}{T} \times \exp\left[\delta_{00}(s_a) + \left(\mathrm{i}s_a + \frac{1}{2}\right)aLz + \psi(s_a)Lz + \frac{(-\ln T - aLz)^2}{2\psi''(s_a)Lz}\right]. \tag{22}$$

In the above equation, the function $\delta_{00}(s)$ is defined by

$$\exp\left[\delta_{00}(s)\right] = \frac{\Gamma(1+\mathrm{i}s)2^{2\mathrm{i}s}\tanh\pi s}{2i\pi^{1/2}\Gamma\left(\frac{1}{2}+\mathrm{i}s\right)}\langle 0|\lambda^{(0)}(s)\rangle\langle\lambda^{(0)}(s)|0\rangle \tag{23}$$

s_a by

$$s_a = \mathrm{i}/2 + \delta_{00}'(s_a)/\psi''(s_a)Lz \tag{24}$$

and the centroid of the distribution, aLz, is found from

$$\psi'(s_a) = -\mathrm{i}a. \tag{25}$$

Since the position of the saddle is very close to $s = \mathrm{i}/2$. the replacements $\delta_{00}(s_a) \simeq \delta_{00}(\mathrm{i}/2)$ and $\psi''(s_a) \simeq \psi''(\mathrm{i}/2)$ are justified in equation (22). It is easily found that, irrespective of the form of $p_1(\mathbf{M})$, one has

$$\exp[\delta_{00}(\mathrm{i}/2)] = 1/2\pi \qquad \psi(s_a) \simeq -\mathrm{i}a\delta'_{00}(s_a)/\psi''(s_a)Lz. \tag{26}$$

It is also found that $\psi''(\mathrm{i}/2)$ is real and negative.

The contribution to the integral $I_{00}^{C^0}$ stemming from $\omega_{00}^{(+\,-)}(s)$, and the portion of the contour for which the decomposition (21) is not valid (if such a portion exists) will decay with Lz relative to the contribution due to $\omega_{00}^{(0)}(s)$. It will be assumed that no poles exist between the real-axis and the deformed path, so that the dominant contribution to the integral $I_{00}^{C^0}$ is that of the saddle. From (22), (26) and the negative nature of $\psi''(\mathrm{i}/2)$, one thus obtains the following form for the probability density of $-\ln T$:

$$\begin{aligned} p_{1\to Lz}(-\ln T)\,\mathrm{d}(-\ln T) &= I_{00}^{C^0}(T, Lz)\frac{\mathrm{d}(-\ln T)}{T} \\ &= \frac{1}{\sqrt{2\pi bLz}}\exp\left[-\frac{(-\ln T - aLz)^2}{2bLz}\right]\mathrm{d}(-\ln T) \end{aligned} \tag{27}$$

with

$$\begin{cases} a = \mathrm{i}\psi'(\mathrm{i}/2) \\ b = -\psi''(\mathrm{i}/2). \end{cases} \tag{28}$$

This is of precisely the same form as the random-phase solution; the extension to non-circular ensembles merely induces a more complicated form for the function $\psi(s)$ (being derived from the eigenvalue of greatest modulus of an infinite matrix).

Attention will now be turned to the extraction of the $I_{mm'}^{C^0}$ for general m, m'. It is found that $\omega_{mm'}^{(0)}(s)$ possesses a saddle point close to $s = \mathrm{i}/2$ for all mm'. The integration contour will thus be deformed to pass through the saddle as in the evaluation of $I_{00}^{C^0}$ discussed previously. The saddle yields the following contribution to the integral $I_{-m-m'}^{C^0}$:

$$I_{-m-m'}^{(0)C^0}(\mathbf{M}, Lz) = \exp[\mathrm{i}(2m\mu + 2m'\nu)]I_{00}^{C^0}(\mathbf{M}, Lz)g_m^{\mathrm{r}}\; g_{m'}^{\mathrm{l}} \tag{29}$$

where

$$\begin{aligned} &g_m^{\mathrm{r}} = \langle -m|\lambda_{\mathrm{m}}(\mathrm{i}/2)\rangle \\ &\begin{cases} g_{m'}^{\mathrm{l}} = (-1)^{m'}|m'|\lim_{s\to\mathrm{i}/2}\left[\langle\lambda_{\mathrm{m}}(s)| - m'\rangle\left(\tfrac{1}{2} + \mathrm{i}s\right)^{-1}\right] & m' \neq 0 \\ g_0^{\mathrm{l}} = \langle\lambda_{\mathrm{m}}(\mathrm{i}/2)|0\rangle. \end{cases} \end{aligned} \tag{30}$$

Consistent with the normalization condition, it is possible and convenient to make the choice $\langle 0|\lambda_{\mathrm{m}}(\mathrm{i}/2)\rangle = \langle\lambda_{\mathrm{m}}(\mathrm{i}/2)|0\rangle = 1$, implying in particular that $g_0^{\mathrm{r}} = g_0^{\mathrm{l}} = 1$. The similarity transformation (14) enables the $\{g_{m'}^{\mathrm{l}}\}$ to be expressed more simply by

$$g_{m'}^{\mathrm{l}} = (-1)^{m'}\langle -m|\lambda_{\mathrm{m}}(-\mathrm{i}/2)\rangle. \tag{31}$$

In contrast to the situation in the evaluation of $I_{00}^{C^0}$, for m, m' both positive or both negative, there exists a (simple) pole at $s = \mathrm{i}/2$ of $\exp[\omega_{mm'}^{(+\,-)}(s)]$. This pole will contribute to the integral, after the deformation of the contour, an amount which is calculated to be

$$\begin{cases} -(m'/m)\hat{\chi}^{2(0)}_{mm'}\left(\langle\hat{\chi}^{2(0)}_{-}\rangle^{Lz}\right)_{-m-m'} & \text{if } m, m' \text{ both positive} \\ -(m'/m)\hat{\chi}^{2(0)}_{mm'}\left(\langle\hat{\chi}^{2(0)}_{+}\rangle^{Lz}\right)_{-m-m'} & \text{if } m, m' \text{ both negative.} \end{cases}$$

The matrices $\hat{\chi}^{2(0)}_{+}$ and $\hat{\chi}^{2(0)}_{-}$ are related by a simple similarity transformation to $D^1 \in D^-$ and $D^1 \in D^+$ respectively (this follows by use of (14) and the similarity transformation which relates $\hat{\chi}^{2(-k)}$ to D^k, equation (A.10)). Indeed it is found that the pole contribution acts precisely to cancel the contribution of $D^1 \in D^{\pm}$ in the expansion (3). The contribution of the integration along the deformed path other than that due to the saddle of $\omega^{(0)}_{mm'}(s)$ can be ignored for $Lz \gg 1$.

3.2. *Contribution of the class $C^{1/2}$*

The contribution of the class $C^{1/2}$ in the expansion of the probability density, equation (4), can be evaluated in a manner which is precisely analogous to that presented in the previous subsection for the class C^0. The set of integrals $\{I^{C^{1/2}}_{mm'}(\mathsf{M}, Lz)\}$ $(m, m' = \pm 1/2, \pm 3/2, \ldots)$ will be defined in direct analogy to the $\{I^{C^{1/2}}_{mm'}(\mathsf{M}, Lz)\}$ (equation (7)). As was the case for the continuous class C^0, it is found advantageous to span $C^{1/2}$ with the $\hat{\chi}^{2(-1/2-\mathrm{i}s)}$. Under the assumption that within this class, the eigenspectrum of $\langle\hat{\chi}^{2(-1/2-\mathrm{i}s)}\rangle$ is discrete at its upper limit in the vicinity of $s = \mathrm{i}/2$, the integrals $\{I^{C^{1/2}}_{mm'}\}$ can be performed by the saddle-point procedure (no poles are encountered for these integrals). The result is

$$\begin{aligned} I^{C^{1/2}}_{-m-m'}(\mathsf{M}, Lz) &= \exp[\mathrm{i}(2m'\mu + 2m'\nu)]\ \langle -m|\lambda^{C^{1/2}}_{\mathrm{m}}(\mathrm{i}/2)\rangle\langle\lambda^{C^{1/2}}_{\mathrm{m}}(\mathrm{i}/2)| - m'\rangle \\ &\quad \times (-1)^{\mathrm{int}[|m'|]}|m'|\ (\pi/T)\frac{1}{\sqrt{2\pi b_{C^{1/2}} Lz}} \\ &\quad \times \exp\left[\psi_{C^{1/2}}(\mathrm{i}/2)Lz - \frac{(-\ln T - a_{C^{1/2}}Lz)^2}{2b_{C^{1/2}}Lz}\right]. \end{aligned} \tag{32}$$

The essential observation is that $\psi_{C^{1/2}}(\mathrm{i}/2)$, in contrast to the value $\psi(\mathrm{i}/2)$ pertaining to the C^0 class (which is zero), is a negative quantity. This implies that the contribution of the class $C^{1/2}$ decays exponentially relative to that of the class C^0, so that for $Lz \gg 1$ the contribution of $C^{1/2}$ to the probability distribution $p_{1\to Lz}(\mathsf{M})$ may be neglected. (This dominance can also be deduced if it is the case that the spectrum of $\langle\hat{\chi}^{2(-1/2-\mathrm{i}s)}\rangle$ is continuous at its upper limit, although the exponential nature will in general be modified.)

3.3. *Contribution of the discrete classes $D^{\pm}$*

It is easily deduced that the eigenspectra of the averaged representations belonging to the discrete classes extend to values below unity. Since the $\{D^k_{mm'}\}$ for $k = 3/2, 2, 5/2, \ldots$ are normalizable with respect to the invariant measure, all but the contribution of $D^1 \in D^{\pm}$ in the expansion of the probability density can be ignored for $Lz \gg 1$. However, it has been deduced that the latter contribution is precisely cancelled by a component arising from the continuous class C^0.

3.4. *The resulting distribution*

The result of the previous three subsections is that the probability density $p_{1\to Lz}(\mathsf{M})$ is given simply by $\sum_{mm'} I^{(0)C^0}_{mm'}(\mathsf{M}, Lz)$. Insertion of the result (29) for the $\{I^{(0)C^0}_{mm'}\}$ yields the following factorized form for the distribution

$$p_{1\to Lz}(\mathsf{M})\,\mathrm{d}\mu(\mathsf{M}) = p^{(T)}_{1\to Lz}(T)p^{(\mu)}_{\mathrm{lim}}(\mu)p^{(\nu)}_{\mathrm{lim}}(\nu)\,\mathrm{d}T\,\mathrm{d}\mu\,\mathrm{d}\nu \tag{33}$$

with

$$p^{(T)}_{1\to Lz}(T) = \frac{1}{\sqrt{2\pi bLz}}\frac{1}{T}\exp\left[-\frac{(-\ln T - aLz)^2}{2bLz}\right] \tag{34a}$$

$$p^{(\mu)}_{\mathrm{lim}}(\mu) = \frac{1}{2\pi}\left\{1 + 2\mathrm{Re}\left[\sum_{m=1}^{\infty} g^{\mathrm{r}}_m \exp(2\mathrm{i}m\mu)\right]\right\} \tag{34b}$$

$$p^{(\nu)}_{\mathrm{lim}}(\nu) = \frac{1}{2\pi}\left\{1 + 2\mathrm{Re}\left[\sum_{m'=1}^{\infty} g^{\mathrm{l}}_{m'} \exp(2\mathrm{i}m'\nu)\right]\right\}. \tag{34c}$$

The distributions $p^{(T)}_{1\to Lz}(T)$, $p^{(\mu)}_{\mathrm{lim}}(\mu)$ and $p^{(\nu)}_{\mathrm{lim}}(\nu)$ are separately normalized with respect to integration over the range of their arguments.

The transformation of variables from the phases μ and ν to the reflection and transmission phases (ϕ_{r} and ϕ_{t}) is straightforward: the joint distribution of the latter two variables is given by

$$p^{(\phi_{\mathrm{r}},\phi_{\mathrm{t}})}_{\mathrm{lim}}(\phi_{\mathrm{r}}, \phi_{\mathrm{t}}) = p^{(\mu)}_{\mathrm{lim}}\left(\phi_{\mathrm{t}} - \phi_{\mathrm{r}}/2\right) p^{(\nu)}_{\mathrm{lim}}\left(\phi_{\mathrm{r}}/2\right). \tag{35}$$

The asymptotic distribution of the single variable ϕ_{r}, which in the present work is given simply by $\int p^{(\phi_{\mathrm{r}},\phi_{\mathrm{t}})}_{\mathrm{lim}}(\phi_{\mathrm{r}}, \phi_{\mathrm{t}})\mathrm{d}\phi_{\mathrm{t}} = p^{(\nu)}_{\mathrm{lim}}(\phi_{\mathrm{r}}/2)$, has been studied earlier [15] by an approach which can be shown to be in exact correspondence with the evaluation of (34*c*) with the argument $\phi_{\mathrm{r}}/2$.

It is apparent that the bulk of the weight of the probability density $p_{1\to Lz}(\mathsf{M})$ is given purely in terms of the eigenvalue of greatest modulus and the associated left and right eigenvectors of the averaged representation $\langle\hat{\chi}^{2(-1/2-\mathrm{i}s)}\rangle$ in the close vicinity of $s = \mathrm{i}/2$. With respect to the distributions appearing in the factorization (33) of the distribution, $p^{(T)}_{1\to Lz}(T)$ is determined purely by the eigenvalue information, $p^{(\mu)}_{\mathrm{lim}}(\mu)$ is just the Fourier transform of the right eigenvector at $s = \mathrm{i}/2$ and $p^{(\nu)}_{\mathrm{lim}}(\nu)$ is the Fourier inverse of the left eigenvector (after the relevant limiting procedure of equation (30)). The non-trivial nature of these eigenvectors inherent in the description of non-circular models implies that for such ensembles the phase distributions do not become flat even in the limit of large Lz. The intricate co-dependence of the matrix elements of $\langle\hat{\chi}^{2(-1/2-\mathrm{i}s)}\rangle$ on the form of the distribution $p_1(\mathsf{M})$ of the individual scatterers and on s indicates that, in general, single parameter scaling of $p^{(T)}_{1\to Lz}(T)$ will not be obeyed.

Before applying the foregoing analysis to particular models, it is worthwhile refining the treatment of $p^{(T)}_{1\to Lz}(T)$. In particular, as well as the mean and variance, which are sufficient to describe the bulk of the distribution, all the higher-cumulants of the distribution of $-\ln T$ can be determined from the properties of $\langle\hat{\chi}^{2(-1/2-\mathrm{i}s)}\rangle$ in the neighbourhood of $s = \mathrm{i}/2$. These higher cumulants characterize the deviations from the Gaussian form of the tails of the distribution.

4. The cumulants of $-\ln T$

The statistical cumulants of $-\ln T$, which can be used to characterize the probability distribution of that variable, can be determined from the behaviour of the moments $\langle T^\nu \rangle$ in the region of ν close to zero. For a system consisting of Lz scatterers, one has explicitly

$$c_n^{1\to Lz}(-\ln T) = \frac{\mathrm{d}^n}{\mathrm{d}(-\nu)^n} \ln \langle T^\nu \rangle_{Lz} \bigg|_{\nu=0}. \tag{36}$$

The dependence of the moments as a function of ν and Lz can be extracted by direct expansion in terms of the irreducible representations; since these quantities have no phase dependence, only $\hat{\chi}_{00}^{2(-1/2-\mathrm{i}s)} \in C^0$ appears in the expansion:

$$T^\nu = \int_0^\infty a_\nu(s) \hat{\chi}_{00}^{2(-1/2-\mathrm{i}s)} \mathrm{d}s. \tag{37}$$

If one attempts to extract the functional form of $a_\nu(s)$ by direct use of the orthogonality relationship (A.7), it is found that the resulting integral is convergent only for $\nu > 1/2$ [11]. The case of $\nu < 1/2$ can be handled by the introduction of discrete components to the harmonic analysis. With the expansion

$$T^\nu = \sum_{p=0}^{n} a_\nu^{(p)} \hat{\chi}_{00}^{-2\nu-p} + \int_0^\infty b_\nu(s) \hat{\chi}_{00}^{2(-1/2-\mathrm{i}s)} \mathrm{d}s \tag{38}$$

it is found that the integral determining the function $b_\nu(s)$ is rendered convergent if the $\{a_\nu^{(p)}\}$ are chosen to be

$$a_\nu^{(p)} = ({}^{-\nu}C_p)^2 \Bigg/ \left[\sum_{j=0}^{\infty} \left(\sum_{k=0}^{p} (-1)^{k-\nu-p} C_{j-k}{}^{j} C_k{}^{-\nu-j} C_{p-k} \right)^2 \right]. \tag{39}$$

In equation (38), the upper limit of the summation is determined by

$$n = \mathrm{int}[-\nu + 3/2] \tag{40}$$

where the function $\mathrm{int}[x]$ yields the largest integer not greater than x.

For ν equal to a negative integer, equations (38) and (39) correspond exactly with a direct symmetry reduction of the direct product of -2ν identical transfer matrices (as presented in [11]). It may thus be regarded as an analytic continuation of this symmetry reduction; at non-negative-integer values of ν the matrices which describe the evolution with length of T^ν become infinite (these are just the representations), and also there enters an integral over a continuous class of such matrices. For $\nu > 1/2$ only this continuous class remains to describe the evolution.

For ν in the region ($\nu < 1/2$), it can be deduced that, for long-lengths, the discrete part corresponding to $p = 0$ in the expansion of $\langle T^\nu \rangle_{Lz}$ dominates over the remaining contributions (this expansion is obtained by simply averaging (38)). In the long-length limit, the latter components become irrelevant, and the evolution of $\langle T^\nu \rangle_{Lz}$ is given simply by

$$\langle T^\nu \rangle_{Lz} = \frac{\Gamma^2(1-\nu)}{\Gamma(1-2\nu)} (\langle \hat{\chi}^{-2\nu} \rangle^{Lz})_{00} \qquad \nu < 1/2. \tag{41}$$

Returning to the evaluation of the cumulants, substituting the explicit form (40) for the moments into equation (36), and making use of the dominance of the eigenvalue of greatest modulus for $Lz \gg 1$ yields (after the change of variables $\nu = 1/2 + \mathrm{i}s$)

$$c_n^{1\to Lz} = \alpha_n Lz + \beta_n \tag{42}$$

where

$$\alpha_n = \mathrm{i}^n \psi^{(n)}(\mathrm{i}/2) \tag{43}$$

and

$$\beta_n = \begin{cases} \mathrm{i}(\mathrm{d}/\mathrm{d}s)\left[\langle 0|\lambda^{(0)}(s)\rangle\langle\lambda^{(0)}(s)|0\rangle\right]|_{s=\mathrm{i}/2} & \text{if } n=1 \\ (-1)^n(n-1)!\zeta(n)(2-2^n) & \\ \quad +\mathrm{i}^n(\mathrm{d}^n/\mathrm{d}s^n)\ln\left[\langle 0|\lambda^{(0)}(s)\rangle\langle\lambda^{(0)}(s)|0\rangle\right]|_{s=\mathrm{i}/2} & \text{if } n\geqslant 2. \end{cases} \tag{44}$$

In the above equation, $\zeta(n)$ is the Riemann zeta function of argument n. From the asymptotic length dependence of the cumulants, equation (42), the corresponding distribution can be found following the method of [8]: this procedure yields precisely the distribution (27) obtained previously (there is the notational correspondence $\alpha_1 \to a$, $\alpha_2 \to b$).

5. Application to the Anderson model

The foregoing analysis will now be applied to the case of the Anderson model for a range of disorders and band-energies. The approach will be to calculate the function $\psi(s)$ (in the neighbourhood of $s = \mathrm{i}/2$) together with the Fourier coefficients $\{g_m^{\mathrm{r}}\}$ and $\{g_m^{\mathrm{l}}\}$ either by perturbation theory, or, for more general disorders, by numerical truncation of the relevant (infinite) matrix.

The Hamiltonian for the Anderson model is of a tight-binding form, and is defined in one-dimension by

$$\hat{H} = \sum_i \epsilon_i |i\rangle\langle i| + \sum_{(ij)} V_{ij}|i\rangle\langle j|. \tag{45}$$

Here ϵ_i is the site-energy associated with the orbital $|i\rangle$ centred at lattice-site i, and the V_{ij} are the hopping elements which connect nearest-neighbour sites only. The disorder in the model enters via randomness in the site-energies (diagonal disorder) and/or in the hopping-integrals (off-diagonal disorder). Only the case of diagonal disorder will be investigated here, with the site-energies within the disordered system taken as statistically independent but equivalent variables, each being described by a probability distribution $p(\epsilon)$ which is chosen to be symmetric about the mean. The hopping terms are taken as being equal to V for all nearest-neighbour pairs. The disordered system is taken as the region $1 \leqslant i \leqslant Lz$, and is embedded between two semi-infinite ordered leads with site energies all equal to the average site-energy within the disordered region.

In the basis of the $\{|i>\}$, the Schrödinger equation for energy E can be written in the form

$$\begin{bmatrix}\phi_{n+1}\\ \phi_n\end{bmatrix} = \mathbf{M}_n^{\mathrm{rs}}\begin{bmatrix}\phi_n\\ \phi_{n-1}\end{bmatrix} \qquad \forall n \tag{46}$$

where $\mathbf{M}_n^{rs}$ is a real-space transfer matrix determined purely by the properties of the nth scatterer, and is given by

$$\mathbf{M}_n^{rs} = \begin{bmatrix} (\epsilon_n - E)/V & -1 \\ 1 & 0 \end{bmatrix}. \tag{47}$$

A change of basis to the set of Bloch-waves of the ordered ensemble-averaged system can be performed, yielding a transfer matrix which relates incoming and outgoing propagating waves in the way described in the introduction. It has the explicit form

$$\mathbf{M}_n = \begin{bmatrix} (1 - i\delta_n)e^{ik} & -i\delta_n e^{ik} \\ i\delta_n e^{-ik} & (1 + i\delta_n)e^{-ik} \end{bmatrix}. \tag{48}$$

Here k is the wave-vector, which is related to the energy by the familiar tight-binding dispersion relation

$$E - \langle\epsilon\rangle = -2V\cos k \tag{49}$$

and δ_n is the statistical variable

$$\delta_n = (\epsilon_n - \langle\epsilon\rangle)/2\sin k. \tag{50}$$

Comparing equations (47) and (1), one can identify the parameters r and t for the nth scatterer as

$$r_n = -i\delta_n/(1 - i\delta_n) \qquad t_n = (1/(1 - i\delta_n))\, e^{-ik}. \tag{51}$$

5.1. Weak disorder

Using the Anderson model forms for the reflection and transmission coefficients (equation (51)), the elements of the averaged representation $\langle\hat{\chi}^{2(-1/2-is)}\rangle$ can be calculated as an expansion in the moments (about the mean) of the site-energy distribution $(\langle(\epsilon - \langle\epsilon\rangle)^{2n}\rangle)$, or equivalently in the moments of the distribution $p(\delta)$ of the δ_n (only even moments occur in the expansion since the distribution of the site-energies is assumed symmetric about the mean). Perturbation theory in the disorder can then be applied to the matrices to determine the eigenvalues and eigenvectors. As found for the representation studied in [15] (referred to in that work as the reflection transfer matrix), the nature of the perturbation theory necessary depends on the value of k; if k is rationally related to π by

$$k = p\pi/q \tag{52}$$

non-degenerate perturbation theory can be applied up to $2m$th order in the site-energies where m is given by $m = \mathrm{int}[(q+1)/2]$. At higher-orders the degeneracy amongst the unperturbed zero-disorder eigenvalues needs to be explicitly considered. The strongest effect of this degeneracy thus occurs at the band centre.

5.1.1. Generic band energy. At a position in the band where the use of ordinary non-degenerate perturbation theory is permitted, the eigenvalue of greatest modulus and the associated eigenvectors are easily calculated. Considering first the distribution $p_{1\to Lz}(-\ln T)$ (which is related to $p^{(T)}_{1\to Lz}(T)$ by a simple change of variable), the coefficients α_n and β_n which, together with (42) determine the cumulants of $-\ln T$ for large Lz, are calculated to be

$$\begin{cases} \alpha_1 = \sigma^{(1)} - \frac{1}{2}\sigma^{(2)} + \frac{3}{2}(\sigma^{(1)})^2 + \mathcal{O}(\sigma^{(3)}) & \beta_1 = \mathcal{O}(\sigma^{(2)}) \\ \alpha_2 = 2\sigma^{(1)} - \frac{1}{2}\sigma^{(2)} + \frac{3}{2}(\sigma^{(1)})^2 + \mathcal{O}(\sigma^{(3)}) & \beta_2 = -\pi^2/3 + \mathcal{O}(\sigma^{(2)}) \\ \alpha_3 = 3\sigma^{(2)} - 9(\sigma^{(1)})^2 + \mathcal{O}(\sigma^{(3)}) & \beta_3 = 12\zeta(3) + \mathcal{O}(\sigma^{(2)}) \\ \alpha_4 = 6\sigma^{(2)} - 18(\sigma^{(1)})^2 + \mathcal{O}(\sigma^{(3)}) & \beta_4 = -14\pi^4/15 + \mathcal{O}(\sigma^{(2)}). \end{cases} \tag{53}$$

Here $\sigma^{(n)}$ is the $2n$th moment of δ:

$$\sigma^{(n)} = \langle \delta^{2n} \rangle. \tag{54}$$

For the α_n $(n \geqslant 5)$ one must look to sixth order or higher for the leading contribution. At second order, it is seen that the variance (the second cumulant) is equal to twice the mean (the first cumulant) as $Lz \to \infty$, and that all other cumulants are zero. This result was found in [8] by the use of a highly reducible representation, and implies that the moments $\langle T^{\nu}_{Lz} \rangle$ for $\nu < 1/2$ (apart from a length-independent prefactor) are given correctly by use of the log-normal form which describes the bulk of the distribution.

Figure 1 shows a comparison of the analytical forms for the first three cumulants with a numerical simulation. The k-value was chosen as $\cos^{-1} 0.375$, and the site-energy distribution was taken as flat with variance 0.05 V^2.

The relevant eigenvectors are also easily calculated by perturbation theory, and yield for the joint phase-distribution

$$\begin{aligned} p^{(\phi_r,\phi_t)}_{\text{lim}}(\phi_r, \phi_t) = \frac{1}{(2\pi)^2} \Bigg\{ 1 + \sigma^{(1)} \Bigg[&\frac{\sin(k + 2\phi_t - \phi_r)}{\sin k} \\ &+ \frac{\sin(2k + 4\phi_t - 2\phi_r)}{\sin 2k} \Bigg] + \mathcal{O}(\sigma^{(2)}) \Bigg\} \\ \times \Bigg\{ 1 + \sigma^{(1)} \Bigg[-&\frac{\sin(-k + \phi_r)}{\sin k} + \frac{\sin(-2k + 2\phi_r)}{\sin 2k} \Bigg] + \mathcal{O}(\sigma^{(2)}) \Bigg\}. \end{aligned} \tag{55}$$

The limiting distribution of the transmission phase $p^{(\phi_t)}_{\text{lim}}(\phi_t) = \int p^{(\phi_r,\phi_t)}_{\text{lim}}(\phi_r, \phi_t)\, d\phi_r$ is found from (55) to be given by

$$p^{(\phi_t)}_{\text{lim}}(\phi_t) = \frac{1}{2\pi} \left\{ 1 + \left(\sigma^{(1)}\right)^2 \left[\frac{\cos(2\phi_t)}{2\sin^2 k} - \frac{\cos(4\phi_t)}{2\sin^2 2k} \right] + \mathcal{O}(\sigma^{(3)}) \right\}. \tag{56}$$

Hence to lowest order in the disorder $(\sigma^{(1)})$, $p^{(\phi_t)}_{\text{lim}}(\phi_t)$ is flat over its range.

5.2. Band-centre anomaly

At the band-centre, the unperturbed eigenvalues (pertaining to the ordered system) take only the values ± 1. At lowest order in the disorder, it is possible to work within the subspace of the (infinitely degenerate) unperturbed eigenvalue at unity. This yields a tri-diagonal form for the averaged representations. Numerical truncation and calculation of the relevant eigenvalue and eigenvectors allows for the probability distributions to be determined: the truncation size is increased until convergence is achieved.

The α_n which determine the asymptotic length-dependence of the cumulants (via equation (42)) are calculated to be

$$\alpha_1 = 0.913\,893\,2\sigma^{(1)} \qquad \alpha_2 = 1.914\,053\,3\sigma^{(1)} \qquad \alpha_3 = 0.519\,054\sigma^{(1)}$$
$$\alpha_4 = 1.0493\sigma^{(1)} \qquad \alpha_5 = 0.020\sigma^{(1)}. \tag{57}$$

In contrast to the situation at a generic band-energy, the α_n for $n \geqslant 3$ now *are* of order $\sigma^{(1)}$; the third and higher cumulants thus show a marked anomaly at the band centre at weak disorder. This behaviour implies that the moments $\langle T^\nu \rangle$ ($\nu < 1/2$) will not be calculated correctly by use of the log-normal form which describes the bulk of the weight of the probability distribution in the long-length limit. It is also seen from (57) that the relationship between the mean and variance which exists at a generic band-energy ($\alpha_2 = 2\alpha_1$) is modified at the band-centre. This implies a breakdown of the single-parameter scaling of the distribution of T even in the limit of weak disorder for the Anderson model.

The band-centre anomaly in the first cumulant (or equivalently the localization length) has been observed in references [20–22].

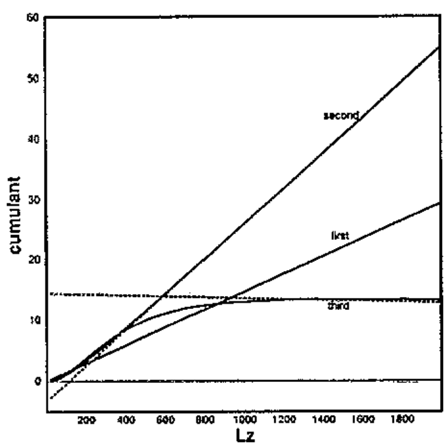

Figure 1. Numerical simulation showing the evolution of the first three cumulants of $-\ln T$ with length, for an Anderson tight-binding Hamiltonian with a flat site-energy distribution of variance 0.05 V^2. The k-value was chosen as $\cos^{-1} 0.375$. The theoretical asymptotic behaviour of the cumulants is included.

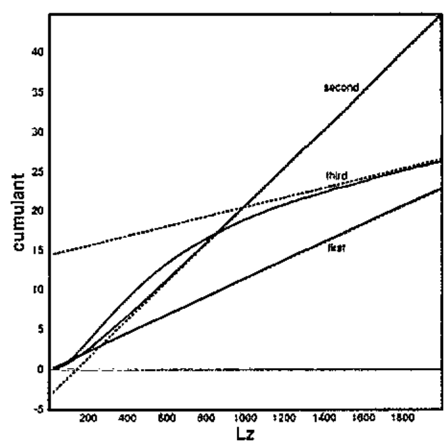

Figure 2. Numerical simulation showing the evolution of the first three cumulants of $-\ln T$ with length, for an Anderson tight-binding Hamiltonian with a flat site-energy distribution of variance 0.05 V^2 at the band-centre. The theoretical asymptotic behaviour of the cumulants is included.

Figure 2 shows band-centre data for the first three cumulants obtained from a numerical simulation, together with the analytical results. The site-energy distribution was again taken as flat with a variance of 0.05 V^2.

The Fourier-components of the distributions of μ and ν are found from the eigenvector information to be

$$
\begin{aligned}
&g_0^{\mathrm{r}} = g_0^{\mathrm{l}} = 1.0000 \qquad g_2^{\mathrm{r}} = g_2^{\mathrm{l}} = -8.6107 \times 10^{-2} \\
&g_4^{\mathrm{r}} = g_4^{\mathrm{l}} = 1.1094 \times 10^{-2} \qquad g_6^{\mathrm{r}} = g_6^{\mathrm{l}} = -1.5872 \times 10^{-2} \ldots .
\end{aligned}
\tag{58}
$$

Note that these values are at *zero-order* in the disorder. The resulting joint distribution of ϕ_{r} and ϕ_{t} is shown in figure 3, and is indistinguishable from the results of a numerical simulation for very small disorders. The reflection-phase distribution at the band-centre has been displayed earlier in [15].

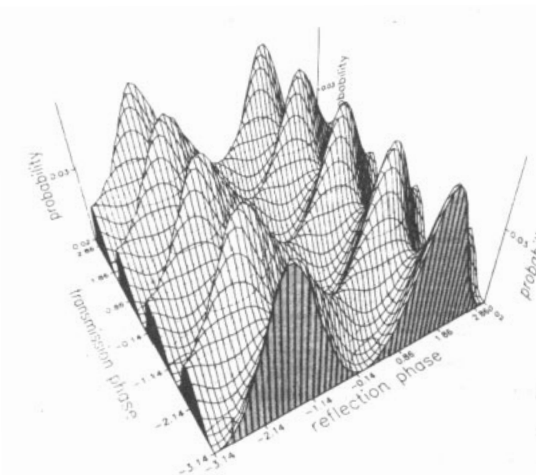

Figure 3. Asymptotic joint distribution of the reflection and transmission phases for the Anderson model at the band-centre and at weak disorder. The plot was obtained by truncation of the representations.

5.3. *Intermediate disorder*

Higher disorders can be dealt with by numerically integrating the elements of $\hat{\chi}^{2(-1/2-is)}$ (which is of course truncated to finite size) weighted by the required probability distribution. The relevant eigenvalue and eigenvectors can then be found numerically, and the truncation size increased until convergence is evident. The range of disorders for which a converged probability distribution can be found by such a method depends on the shape of the site-energy distribution and on the energy: as an example, for a flat site-energy distribution, convergence over most of the band is easily achieved up to a site-energy variance of about 5 V^2. With regard to the phase distributions, it is clearly not possible to calculate the Fourier coefficients ($\{g_m^{\mathrm{r}}\}$ and $\{g_m^{\mathrm{l}}\}$) for arbitrarily large m, implying a lack of knowledge of the high-frequency components of the distributions. In the case of ensembles with continuous disorder distributions, this causes no hindrance since the phase distributions for such models are sufficiently smooth and slowly varying. For discrete distributions, however, the singularities [16] which can occur in the phase distributions will of course not be reproduced, but the approach is sufficiently convergent and numerically stable to produce good overall resolution. Despite the non-negligible contributions of the high-index eigenvector components with respect to the phase-distributions possible in discrete ensembles, it appears that the distribution $p^{(T)}_{1\to L_z}(T)$ stemming from the eigenvalue information displays good convergence as a function of truncation size.

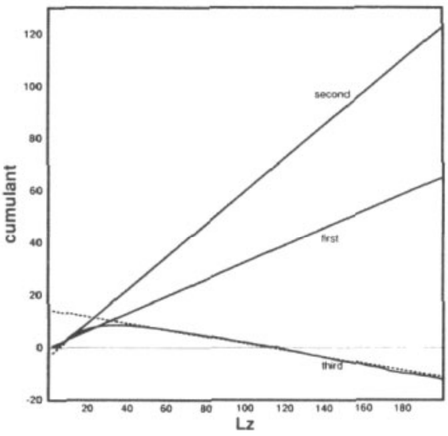

Figure 4. Numerical simulation showing the evolution of the first three cumulants of $-\ln T$ with length, for the Anderson model with a flat site-energy distribution of variance 1.0 V^2. The k-value was chosen as $\pi/3$. The theoretical asymptotic behaviour of the cumulants is included.

Figure 4 shows the length behaviour obtained from a simulation of the first three cumulants at $k = \pi/3$, for a flat distribution of site energies of variance 1.0 V^2. The theoretical asymptotic predictions are included.

The theoretical asymptotic joint reflection and transmission phase distribution for a flat distribution of site energies of variance 1.0 V^2 at $k = \cos^{-1} 0.375$ is shown in figure 5; this distribution is confirmed by numerical experiment. Figure 6 shows the reflection and transmission phase distributions for the binary-symmetric site-energy distribution $p(\epsilon) = 1/2\delta(\epsilon - \epsilon_0 - 1.0) + 1/2\delta(\epsilon - \epsilon_0 + 1.0)$ at $k = 0.45\pi$. Convergence of the first 100 Fourier coefficients was achieved for these distributions. The results agree with numerical simulation at the resolution implied by this truncation ([17] contains a numerical simulation of the reflection distribution for the same ensemble, and provides a discussion of some of the singularities that occur).

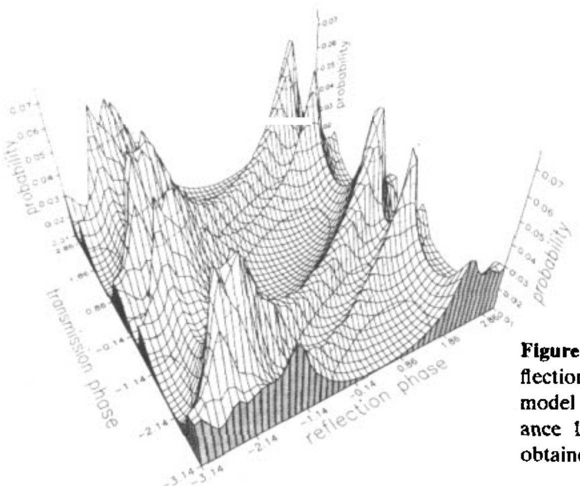

Figure 5. Asymptotic joint distribution of the reflection and transmission phases for the Anderson model with a flat site-energy distribution of variance 1.0 V^2 at $k = \cos^{-1} 0.375$. The plot was obtained by truncation of the representations.

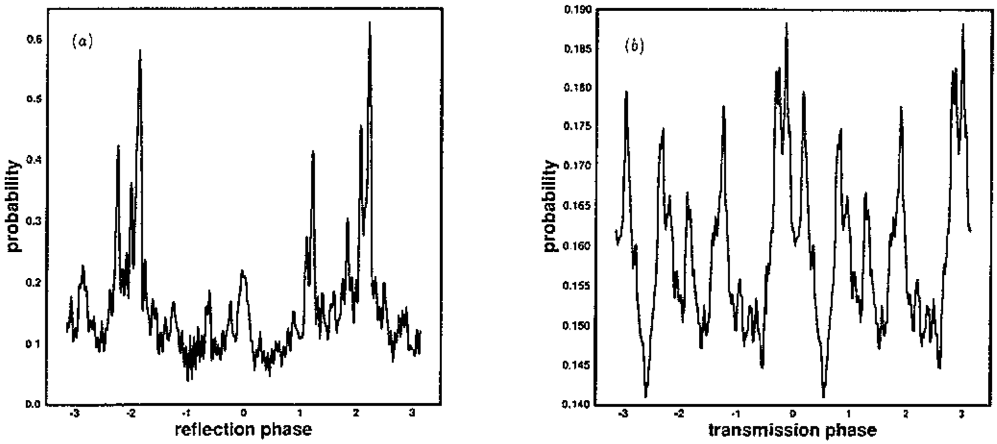

Figure 6. Asymptotic distributions of (*a*) the reflection phase and (*b*) the transmission phase for the Anderson model with the binary-symmetric site-energy distribution $p(\epsilon) = 1/2\delta(\epsilon - \epsilon_0 - 1.0) + 1/2\delta(\epsilon - \epsilon_0 + 1.0)$ at $k = 0.45\pi$. The curves were obtained by truncation of the representations.

6. The resonance tail of the distribution

The form of the probability distribution $p_{1\to Lz}(\mathsf{M})$ with T such that $-\ln T \ll Lz$ can also be evaluated by use of (3) for general $p_1(\mathsf{M})$ by a simple generalization of the method used by Mello [7] for the special case of $p_1(\mathsf{M})$ isotropic over the phase variables. It is found that the contributions of the classes $C^{1/2}$, $D^{\pm}$ decay with Lz relative to the contribution of C^0. With the assumption of the presence of a discrete eigenvalue at the top of the spectrum of $\langle\hat\chi^{2(-1/2-is)}\rangle$ for s close to zero, the set of integrals $\{I^{C^0}_{mm'}\}$ (which are defined as in equation (7)) can be calculated by the saddle-point method following [7] to yield

$$I^{C^0}_{-m-m'}(\mathsf{M}, Lz) = \exp[\psi(0)Lz]\,[\pi/-\psi''(0)]^{3/2} \times \langle -m|\lambda^{(0)}(0)\rangle\langle\lambda^{(0)}(0)|-m'\rangle\hat\chi^{2(-1/2)}_{mm'}(\mathsf{M}). \tag{59}$$

The contribution of the remainder of the eigenvalues of $\langle\hat\chi^{2(-1/2)}_{mm'}\rangle$ can be neglected. The resulting probability density is thus given simply as the sum of the right-hand side of equation (59) over m and m'. The shape of this tail of the distribution is seen to be independent of length; only its overall scale is set by Lz (together with the range of T for which (59) is valid). The eigenvalue and associated eigenvectors appearing in (59) can be calculated by perturbation theory or by numerical truncation. The latter procedure was carried out for the case of the Anderson model; the spectrum was confirmed as being discrete at its upper limit in modulus and the extraction of the eigenvalue of greatest modulus and the associated eigenvectors was found to be convergent and numerically stable over a wide range of disorders and energies (see also [11]).

The distribution of the single variable T within the tail is given simply by

$$p^{\rm res}_{1\to Lz}(T) = I^{C^0}_{00}(T, Lz)\frac{1}{T^2}$$

$$= \exp[\psi(0)Lz]\left[\frac{\pi}{-\psi''(0)}\right]^{3/2} \times \langle 0|\lambda^{(0)}(0)\rangle\langle\lambda^{(0)}(0)|0\rangle T^{-3/2}{}_2F_1\left[\tfrac{1}{2},\tfrac{1}{2},1,1-T\right]. \qquad (60)$$

The moments $\langle T^\nu\rangle$ ($\nu > 1/2$) are dominated by the contribution of this tail of the distribution. Indeed the integrals $\int p^{\rm res}_{1\to Lz} T^\nu\,{\rm d}T$ can be found in the standard tables (e.g. [23]), and are found to be in exact agreement with the $\langle T^\nu\rangle$ ($\nu > 1/2$) obtained in [11] by direct spectral analysis of these quantities.

7. Conclusion

The problem of evaluating the joint probability distribution $p_{1\to Lz}(T,\phi_{\rm r},\phi_{\rm t})$ of transmission intensity, reflection phase and transmission phase of a disordered 1D wire composed of Lz elastic scatterers, given the distribution $p_1(T,\phi_{\rm r},\phi_{\rm t})$ of the constituent scatterers, can be addressed by direct expansion in terms of the irreducible representations of SU(1,1) (the group topologically formed by the (k-space) transfer matrices) without recourse to the assumption of $p_1(T,\phi_{\rm r},\phi_{\rm t})$ isotropic over its phase variables or to the limit of weak disorder. The method was applied to the case of the Anderson model over a range of disorders and band-energies. As well as the bulk of the distribution, the resonance tail can also be determined for a wide range of ensembles.

Acknowledgments

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Appendix 1. The irreducible representations of SU(1,1)

This appendix provides a brief discussion of the unitary irreducible representations of SU(1,1) as categorized originally by Bargmann [17], and introduced in connection with transport in disordered 1D systems by Kirkman and Pendry [11], and Mello [7].

The index of a representation will be denoted k, and the row by m. The representations which form a complete set are listed below.

(i) continuous class:
- (1) integral case (C_k^0), $m = 0, \pm1, \pm2, \ldots$
  $k = 1/2 + {\rm i}s \qquad 0 \leqslant s < \infty$
- (2) half-integral case ($C_k^{1/2}$), $m = \pm1/2, \pm3/2, \ldots$
  $k = 1/2 + {\rm i}s \qquad 0 < s < \infty$

(i) discrete class:
- (1) maximal $m(D_k^-)$
  $k = 1, 3/2, 2, \ldots \qquad m = -k, -(k+1), \ldots$
- (2) minimal $m(D_k^+)$
  $k = 1, 3/2, 2, \ldots \qquad m = k, k+1, \ldots.$

For the display of the explicit forms of the representations, it is convenient to define the variables μ and ν which are related to the reflection and transmission phases, ϕ_r and ϕ_t respectively, by

$$\exp[\mathrm{i}(\mu+\nu)] = \exp[\mathrm{i}\phi_t] \qquad \exp[\mathrm{i}(\mu-\nu)] = \exp[\mathrm{i}(\phi_t-\phi_r)] \tag{A.1}$$

and to define T by

$$T = |\mathrm{t}|^2 = 1 - |\mathrm{r}|^2 \tag{A.2}$$

where r and t are respectively the reflection and transmission amplitudes. The group is spanned by the parameter ranges $0 < T \leqslant 1$, $-\pi \leqslant \mu, \nu < \pi$. In terms of these variables, Bargmann's unitary irreducible representations take the form

$$D^k_{mm'}(\mu, T, \nu) = \mathrm{e}^{-2\mathrm{i}m\mu} d^k_{mm'}(T)\, \mathrm{e}^{-2\mathrm{i}m'\nu} \tag{A.3}$$

with

$$d^k_{mm'}(T) = \Theta_{mm'}(k)(1-T)^{|m-m'|/2} T^k \times {}_2F_1[k+\max(m,m'), k-\min(m,m'), |m-m'|+1; 1-T]. \tag{A.4}$$

In this equation, ${}_2F_1$ is the hypergeometric function of Gauss, and $\Theta_{mm'}$ is defined by

$$\Theta_{mm}(k) = 1 \tag{A.5}$$

$$\Theta_{mm'}(k) = \frac{1}{|m-m'|!} \prod_{j=1}^{|m-m'|} \{k(1-k) + [\min(m,m')+j][\min(m,m')+j-1]\}^{1/2} \times \begin{cases} 1 & m > m' \\ (-1)^{m'-m} & m < m'. \end{cases} \tag{A.6}$$

It can be shown that there exists orthogonality relations amongst the representation elements. This is expressed by

$$\int D^{k*}_{m_1 m_1'}(\mathsf{M}) D^{k'}_{m_2 m_2'}(\mathsf{M})\, \mathrm{d}\mu(\mathsf{M}) = \begin{cases} w_k^{-1}\delta_{kk'}\delta_{m_1' m_2'} & \text{if } k, k' \in D^{\pm} \\ w^{-1}(s)\delta(s-s')\delta_{m_1 m_2}\delta_{m_1' m_2'} & \begin{cases} \text{if } k = \frac{1}{2}+\mathrm{i}s, k' = \frac{1}{2}+\mathrm{i}s' \in C^0 \\ \text{or } k = \frac{1}{2}+\mathrm{i}s, k' = \frac{1}{2}+\mathrm{i}s' \in C^{1/2} \end{cases} \\ 0 & \text{if } k, k' \in \text{different classes} \end{cases} \tag{A.7}$$

where $k = 1/2 + \mathrm{i}s$ and $k' = 1/2 + \mathrm{i}s'$. The invariant measure $\mathrm{d}\mu(\mathsf{M})$ is given by

$$\mathrm{d}\mu(\mathsf{M}) = (2\pi)^{-2}\, \mathrm{d}\mu\, \mathrm{d}\nu \mathrm{d}T/T^2 \tag{A.8}$$

and the weights w_k and $w(s)$ by

$$\begin{cases} w_k = 2k-1 & k \in D^{\pm} \quad k > 1/2 \\ w(s) = 2s \tanh \pi s & k \in C^0 \quad k = 1/2 + \mathrm{i}s \\ w(s) = 2s \coth \pi s & k \in C^{1/2} \quad k = 1/2 + \mathrm{i}s. \end{cases} \tag{A.9}$$

In [11], irreducible representations were derived by decomposing the direct product of n identical transfer matrices according to the permutation group of order n. Within a subspace, a procedure for the analytical continuation to non-positive-integer n was developed. The representations thus constructed are denoted in [11] by $\hat{\chi}^{2N}$. The representation $\hat{\chi}^{2(-k)}$ can be related to D^k as defined in (A.3) by a simple similarity transformation; this transformation merely induces the changes

$$m \to -m, \qquad \theta_{mm'}(k) \to \begin{cases} \Gamma(1-k+m)/\Gamma(1-k+m') & \text{if } m \geqslant m' \\ \Gamma(1-k-m)/\Gamma(1-k-m') & \text{if } m < m'. \end{cases} \tag{A.10}$$

Due to the need for analytic continuation, it is found more convenient to span the continuous representation classes C^0 and $C^{1/2}$ with the $\hat{\chi}^{2(-1/2-\mathrm{i}s)}$.

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